Axiomatic Definition of Valid 3D Parcels, potentially in a Space Partition

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Key words: 3D Geometry and Topology, 3D Cadastres and Models, Legal framework for 3D Cadastre

SUMMARY

The definition of a valid 3D parcel must be correct and unambiguous, because an error or ambiguity in the definition of the extent of a property can lead to expensive legal disputes or to problems with handling 3D parcels in the information systems or problems during data transfer between two systems. This paper develops a rigorous axiomatic definition of a 3D parcel, and its relationship with adjoining parcels within a space partition. Since the requirements of different jurisdictions mandate different levels of validation, some of the axioms are identified as optional. For example, a jurisdiction may require that a parcel must be contiguous, while another may not require this. In earlier publications the axioms concerning valid 3D parcels (within a partition) are formulated in natural language. In this paper we will further formalize this by using mathematical expressions. We also want to prove the necessity of all axioms, i.e. is our set of axioms minimal or are they perhaps overlapping? We show that one of the earlier proposed axioms (A4) is implied by axiom A5 (see discussion in section 3.3) and can be omitted. In order to demonstrate the necessity and independence of the remaining set of axioms, a series of test cases is presented. Each case violates a single axiom and passes all other axioms, thus showing that the set of axioms is non-redundant. In addition, real examples of 3D parcels (From Queensland, Australia), are tested against the validation suite.

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1. INTRODUCTION

As the value of land in the urban regions of the world increases, there is a trend towards the subdivision of property rights in 3D. That is to say, rights to land may be replaced by rights to the space above and below the land. As a result, the simple plans of subdivision that are used in defining property rights on the surface of the earth are being replaced by more complex 3D spatial definitions. This trend has been observed in many different countries and jurisdictions around the world (van Oosterom, Stoter, Ploeger, Thompson and Karki 2011).

An important issue in the framing of these definitions is that they must be correct and unambiguous, because an error or ambiguity in the definition of the extent of a property can lead to expensive legal disputes. This paper addresses two problems: 1. the modelling of a single 3D cadastral parcel, and 2. the modelling of a complete 3D spatial partition.

This paper develops a rigorous axiomatic definition of a 3D parcel (spatial unit), and its relationship with adjoining parcels. Since the requirements of different jurisdictions mandate different levels of validation, some of the axioms are identified as optional. For example, a jurisdiction may require that a parcel be contiguous, while another may not. The suggested axioms are based on Gröger and Plümmer (2011), however they have been modified to reflect these cadastral requirements. Our first steps in this direction were described in our work (Thompson and van Oosterom 2012) In both these papers the definitions, axioms and theorems concerning valid 3D objects (within a partition) are formulated in natural language. In this paper we will further formalize this by using mathematical formalism. We also examine the necessity of all axioms, i.e. is our set of axioms minimal or are they perhaps overlapping? A similar question was raised, but not yet answered, in a recent paper on valid 3D topology structures in general (Brugman, Tijssen and van Oosterom 2011).

In general, it is not possible to prove that this or any set of axioms is complete, because it is always possible to find new cases of "unreasonable" parcels that pass all the validation tests, but the set presented here is shown to be a useful, non-redundant set of axioms that can be used to define practical validation tests, and therefore assist in the reliable transfer of 3D parcel data.

The remainder of this paper is organized as follows. Section 2 provides background material, such as explaining the issues with finite precision, the specific requirements for representing 3D parcels, and the nomenclature used in the paper. The axioms for valid 3D parcels are given as mathematical expressions in Section 3. The representation to unbounded objects (to above or below) as defined in the Land Administration Domain Model is discussed in Section 4. Finally, the main conclusions and future research topics are given in Section 5.

2. BACKGROUND

The representation of 3D objects in, for example CAD (Computer Aided Design/Drafting) is not new, and significant work has been done on ensuring that the computer-based model is valid and is a good representation of the real-world object as it exists, or as it is to be constructed. The problem with the cadastral parcel is slightly different. Cadastral parcels are not "real-world" objects, although they may sometimes be associated with them. A cadastral parcel is a theoretical definition of space. One result of this is that the validation rules of cadastral parcels may have differences from the rules for CAD.
The approach taken by Oracle Spatial is described in (Kazar, Kothuri, van Oosterom and Ravada 2008). This provides a clear description of the rules for validating 3D geometries that are to be imported into the Oracle database, but has restrictions on the boundary surface representations that are problematic for cadastral data.
(Gröger and Plümer 2005) give a simple set of axioms that define a "2.8D" coverage – which has many of the required attributes of the problem domain, but with restrictions. One of the restrictions – the inability to model bridges or tunnels has been removed in a later paper (Gröger and Plümmer 2011). This approach, based on the "2.8D" paradigm is also unsuitable for cadastral parcels as requires that boundary surfaces be 2D manifolds (Thompson and van Oosterom 2012).

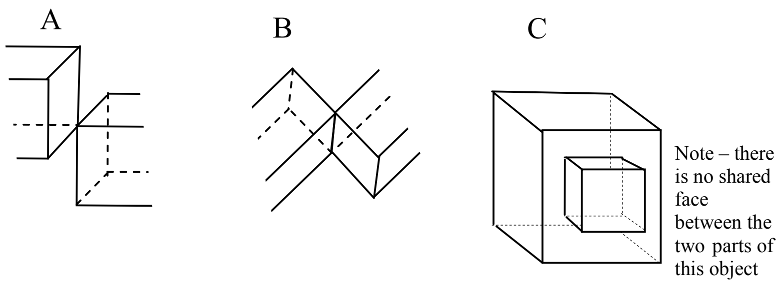


Figure 1. Cases A and B are disallowed by the axioms of Gröger and Plümer. Case C is disallowed by the Oracle Spatial validation rules

2.1 Finite precision

In the 2D world of GIS, the problem of finite precision of computation is often addressed by the process of normalisation (Milenkovic 1988), or some variant. This paper extends the concept into 3D. It is critical to the approach that there exists a tolerance value ε with the characteristic that all arithmetic operations can be assured to give a result that is correct to an order of magnitude smaller that ε (Milenkovic uses 1/10).
This gives rise to a question of point identity. There is a distinction to be made between points which are close together, points at zero distance apart, and points which are identified as the same logically. In this paper, it is assumed that points are uniquely identified (by name), and so the statement that points must be a minimum of ε apart excludes the possibility that two points are at the same location (in x, y and z).

2.2 Cadastral parcels

Although cadastral parcels in 3D have a similarity to 3D objects, at least in terms of their modelling and computer representation, there are significant differences.

1. A cadastral parcel is not physical object, and cannot be seen (although it may contain or even be defined by real world objects).
2. The definition of a cadastral parcel may have points, lines, or faces in common with adjoining parcels, and this fact must be clear in the definition. For example, if in Figure 1, objects A and B represent pairs of cadastral parcels, the sharing of the point/line is significant. By contrast if these objects represent part of a 3D city model for example, the parts can simply be moved apart slightly to allow the rules of Gröger and Plümer to be observed.
3. A cadastral parcel may not be fully bounded. This case arises because in many jurisdictions, the definition of the conventional 2D land parcel has no top or bottom defined. Thus, if a 3D parcel is excised from a 2D parcel, the remaining space may have no upper surface, or lower surface.

2.3 Nomenclature

In the mathematical space ($\mathbf{R}^3$) (Note – this assumes exact arithmetic for the purpose of the definitions).

Let P be the set of all real-number valued points $p = (x,y,z)$.

For $p_1 = (x_1,y_1,z_1)$, $p_2 = (x_2,y_2,z_2)$, $\mathrm{D}(p_1, p_2) =_{\mathrm{def}} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

For $p_1, p_2 \in P$, $p_1 = p_2 =_{\mathrm{def}} \mathrm{D}(p_1, p_2) = 0$;

In what follows, where the context is clear, the definitions of variables are omitted. For example, if p_1 represents a point, the definition $p_1 \in P$ will be omitted.

A **node** $n \in P$ is a special case of point, which can be represented in the number system of the computer (for example as a set of floating point numbers).

Let N be the set of all possible nodes, $N \subset P$.

A **directed-edge** e is an ordered pair of nodes $e = (n_1, n_2)$:

Let E be the set of all possible directed-edges.

For directed edges $e_1 = (n_1, n_2)$, $e_2 = (m_1, m_2) \in E$,

$$e_1 = e_2 =_{\mathrm{def}} n_1 = m_1 \wedge n_2 = m_2$$

$$e_1 \rightleftharpoons e_2 =_{\mathrm{def}} n_1 = m_2 \wedge n_2 = m_1$$

$$\bar{e}_1 =_{\mathrm{def}} (n_2, n_1)$$

The notation is used that $n \in e$ means if $e = (n_1, n_2)$ then $n = n_1$ or $n = n_2$.

$\mathrm{On}(p, e) =_{\mathrm{def}} \exists\ t \in \mathbf{R}$: $0 \leq t \leq 1$, $x = x_2 + t(x_1\text{-}x_2)$, $y = y_2 + t(y_1\text{-}y_2)$, $z = z_2 + t(z_1\text{-}z_2)$. Where $e = (n_1,n_2)$, $n_1 = (x_1,y_1,z_1)$, $n_2 = (x_2,y_2,z_2)$,

For directed-edge e, point p, $\mathrm{D}(p, e) = \min_{On(p_1,e)} \mathrm{D}(p, p_1)$.

For directed-edges e_1, e_2, $\mathrm{D}(e_1,e_2) = \min_{On(p_1,e_1)} \mathrm{D}(p_1,e_2)$.

A **face** f is defined as a set of nodes f_n, a set of directed-edges f_e and a tuple of numbers $f_p = (a,b,c,d)$: $a,b,c,d \in \mathbf{R}$ restricted as follows:

$\forall e = (n_1,n_2) \in f_e$: $n_1, n_2 \in f_n$.
$\forall n \in f_n$: $\{e_1=(n_1,n_2): n_1 = n\}$ and $\{e_2=(n_1,n_2): n_2 = n\}$ are of same cardinality.
$a^2+b^2+c^2 = 1$;

$f =_{def} f(f_n, f_e, f_p)$. Where the context is clear, f will be used to mean f_e, f_n or f_p. e.g. $n \in f$.
The plane for face $f_p = (a,b,c,d)$ is $\{p = (x,y,z) \in P: ax+by+cz+d = 0\}$.
Let F the set of all possible faces.
For any faces defined on the same set of nodes, the plane parameters must agree.

For $f = f(f_n, f_e, f_p), f' = f'(f'_n, f'_e, f'_p), f_n = f'_n \Rightarrow f_p = f'_p \vee f_p = -f'_p$.

This can be a difficult issue, because if the faces' planar parameters are not supplied, it is up to the receiving program to determine them. This means that the algorithm must be repeatable to within the accuracy of the calculation, or that the equality of the node sets must be detected, and the calculation carried out once only.
If this constraint is not respected, many of the tests related to dihedral angles between faces will be complicated and difficult to make consistent.
For e_1, e_2, $n \in f$, $n \in e_1$, $n \in e_2$, let $a(e_1, e_2, n)$ be the angle between e_1 and e_2 at n measured anticlockwise around n, as viewed from outside the face (i.e. from the side of the face for which $ax+by+cz+d > 0$).
For $e_1 \in f_1$, $e_2 \in f_2$, $e_1 = e_2 \vee e_1 \rightleftharpoons e_2$, let $A(f_1, f_2, e_1)$ be the dihedral angle between f_1 and f_2 at e_1 measured anticlockwise around the directed-edge looking in the direction of the edge – so that $A(f_1, f_2, e_1) = -A(f_1, f_2, e_2) = -A(f_2, f_1, e_1) = A(f_2, f_1, e_2)$.

For plane f_p and point $p = (x,y,z)$, $D(p, f_p) = |ax+by+cz+d|$.

$\forall\ n_i \in f$, let n_i' be the point at the base of the normal from n_i to f_p. The points n_i' form a planar multi-polygon. Let $On(p, f)$ be $D(p, f_p) = 0 \wedge$ (p is inside the closure of the planar polygon).
$\forall\ p \in P$, let p' be the point at the base of the normal from p to f_p.

$In(p, f) =_{def} On(p', f_p)$. (that is, the base of the normal is within the face).

A **shell** i s s a set of faces s_f and their associated directed edges s_e and nodes s_n.
$s_f \subseteq F$
$s_e = \{e: \exists f: e \in f \wedge f \in s_f\}$
$s_n = \{n: \exists f: n \in f \wedge f \in s_f\}$

Note – the definition of a shell as a set of faces ensures that two identical faces cannot be within the same shell, but an implementation will have to ensure that $f_1, f_2 \in s \Rightarrow f_1 \neq f_2$. Also note that by this definition a shell may not enclose 3D space. Later the concept of a "cycle shell" will be introduced as a fully bounded shell.

An **edge** $u\ (s_f, n_1, n_2)$ within s_f is defined as:

$u\ (s_f, n_1, n_2) = \{e = (m_1,m_2) \in s_e \wedge m_1 = n_1 \wedge m_2 = n_2\}$
$\quad \cup \{e = (m_1,m_2) \in s_e \wedge m_1 = n_2 \wedge m_2 = n_1\}$.
$s_u = \{u: \forall e \in u, e \in s_e\}$
shell $s =_{def} s_f \cup s_e \cup s_n \cup s_u$ Corollary $u\ (s, n_1, n_2) = u\ (s, n_2, n_1)$, hence u is undirected.

Let the set of edges be U, and let $u \in s$ mean that $\forall e \in u, e \in s$.

A **corner** $v_2(f, e_1, e_2)$ within face is the meeting of two directed edges e_1 and e_2 such that

$\exists\, n \in N$: $(e_1, e_2, n \in f, n \in e_1, n \in e_2)$ and
$e \in f, n \in e \Rightarrow \mathrm{a}(e_1, e_2, n) < \mathrm{a}(e_1, e, n)$

(Descriptively, a corner is a pair of edges meeting at a node, with no intervening edges in the same face. Note - if the edges do not have a node in common, a corner is not defined).
Let V_2 be the set of all corners.
Notation: $e \in v_2$ where $v_2 = v_2(f, e_1, e_2)$, $(v_2 \in V_2)$ is taken to mean $e = e_1$ or $e = e_2$.

A **fold** $v_3(s, f_1, e_1, f_2)$ within shell s is the meeting of two faces f_1 and f_2 at directed edges e_1 and e_2 such that $f_1 \in s$, $f_2 \in s$, $e_1 \in f_1$, $e_2 \in f_2$, $e_1 \rightleftharpoons e_2$ and:

$(f \in s, e \in f, e \rightleftharpoons e_1) \Rightarrow \mathrm{A}(f_1,f_2,e_1) < \mathrm{A}(f_1,f,e_1) \wedge$
$(f \in s, e \in f, e = e_1) \Rightarrow \mathrm{A}(f_1,f_2,e_1) \leq \mathrm{A}(f_1,-f,e_1)$.

Where $-f =_{\mathrm{def}}\ f_n$, $-f_e = \{\bar{e}: e \in f_e\}$, and $(-a,-b,-c,-d)$ – that is the same face but with the reversed sense.
(Descriptively, a fold is a pair of faces that meet at an edge, with no intervening faces at the same edge between them).
Let V_3 be the set of all folds.
Notation: $f \in v_3$ where $v_3 = v_3(s,f_1,e_1,f_2)$, $(v_3 \in V_3)$ is taken to mean $f = f_1$ or $f = f_2$, while $e \in v_3$ is taken to mean $e = e_1$ or $e \rightleftharpoons e_1 \wedge e \in f_2$.

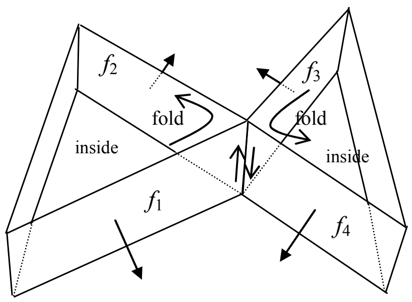


Figure 2. An edge which is the meeting of 4 faces, 2 folds and 4 directed edges

A **C_0 face** $f' = f'(e)$ in shell s is a subset of a face $f \in s$ such that:

$e \in f'$
$f'_p = f_p$
$e_1 \in f' \Rightarrow n_1 \in f' \wedge n_2 \in f'$ (where $e_1 = (n_1, n_2)$)
$n \in f' \Rightarrow (\forall\ \mathrm{e} \in f: n \in e \Rightarrow e \in f')$

(Descriptively, for any edge in f' the nodes that define it are in f', for every node, all directed edges that meet at that node are in f').

$\mathrm{C}_0(f) =_{\mathrm{def}} \exists\ e \in f \wedge f = f'(e)$

(Descriptively, a face is zero-connected if it has at least one directed edge, and all other edges/nodes are zero-connected to it)

A **C_1 face** $f'' = f''(e)$ in shell s is a subset of a face $f \in s$ such that:

$f''_p = f_p$

$e_1 \in f'' \Rightarrow n_1 \in f'' \wedge n_2 \in f''$ (where $e_1 = (n_1, n_2)$)
$e_1 \in f'' \wedge e_2 \in f \wedge v_2(f, e_1, e_2) \in V_2 \Rightarrow e_2 \in f''$
(Descriptively, for any edge in f'' the nodes that define it are in f'', and all directed edges that meet at that node in a corner are in f'').
$C_1(f) =_{def} \exists\, e \in f \wedge f = f''(e)$

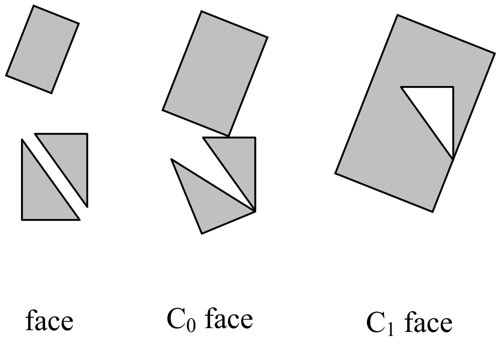


Figure 3. Faces, with weak and strong connectivity depicted

In common parlance, a face and a C_0 face would be referred to as multi-polygons, with a C_1 face being a simple polygon.

A **cycle shell** s is a shell such that:

$\forall u \in s$: for $e \in u$ $\{e_1: e_1 = e\}$ and $\{e_2: e_2 \doteqdot e\}$ are of same cardinality.
(Every undirected edge in the shell is composed of anti-equal pairs of directed edges).
(A cycle shell defines a bound region of space).

A $\mathbf{C_0}$ **shell** $s' = s'(f)$ in cycle shell s a subset of s such that

$C_0(f) \wedge f \in s' \wedge$
$f_1 \in s' \wedge n \in f_1 \wedge f_2 \in s \wedge C_0(f_2) \wedge n \in f_2 \Rightarrow f_2 \in s'$
(For every face in s' all C_0 connected faces that meet it at a node are also in s').
$C_0(s) =_{def} \exists f \in s \wedge s = s'(f)$

A $\mathbf{C_1}$ **shell** $s'' = s''(f)$ in cycle shell s a subset of s such that

$C_1(f) \wedge f \in s' \wedge$
$f_1 \in s'' \wedge e_1 \in f_1 \wedge f_2 \in s \wedge C_1(f_2) \wedge e_2 \in f_2 \wedge (e_1 \doteqdot e_2) \Rightarrow f_2 \in s''$
(For every face in s'' all C_1 connected faces that meet it at any edge are also in s'').
$C_1(s) =_{def} \exists f \in s \wedge s = s''(f)$

A $\mathbf{C_2}$ **shell** $s^\circ = s^\circ(f)$ in cycle shell s a subset of s such that

$C_1(f) \wedge f \in s^\circ \wedge$
$f_1 \in s^\circ \wedge e_1 \in f_1 \wedge f_2 \in s \wedge C_1(f_2) \wedge \exists v_3 (s, f_1, e_1, f_2) \Rightarrow f_2 \in s^\circ$
(For every face in s° all C_1 connected faces that meet it at a fold are also in s°).
$C_2(s) =_{def} \exists f \in s \wedge s = s^\circ(f)$

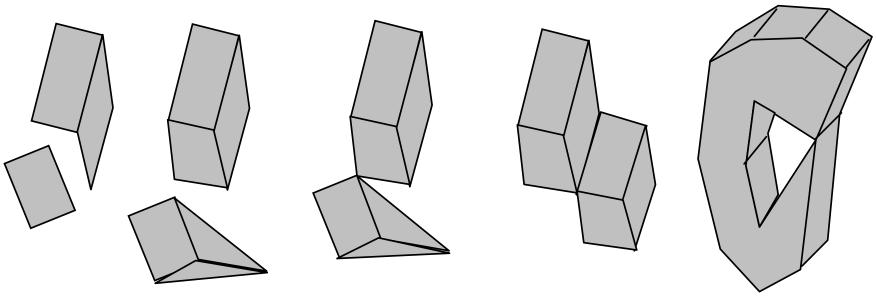


Figure 4. Shell, cycle shell, two types of weak connectivity and strong connectivity

The cycle shell, C_0 shell and C_1 shell can be referred to as multi-polyhedra, while the C_2 shell can be termed a simple polyhedron. Note that any polyhedron can, by this definition, have holes, which may "tunnel through" the body, or be total inclusions with no connection to the outer boundary.

3. THE AXIOMS

An axiom is considered to be necessary if it cannot be derived from other axioms, and some unacceptable consequences follow from its violation. Therefore, an attempt has been made in each case to generate a test case which fails a single axiom (and no other), and an explanation is given as to why that case is problematic. The numbering of the axioms is the same as in the paper (Thompson, van Oosterom, 2012) where these axioms where presented in natural language.

3.1 Core axioms

These axioms are considered to be essential, because non-compliant objects would potentially cause malfunctioning of software that uses the data. All axioms apply to a particular shell *s*.

Axiom A1 No two nodes are closer than ε apart.

$$\text{A1}: n_1, n_2 \in s: (n_1 \neq n_2) \Rightarrow \mathrm{D}(n_1, n_2) > \varepsilon$$

Many test cases are possible, but the one illustrated in Figure 5 does not violate any other axiom.
There are two main unacceptable characteristics of objects which do not satisfy this:

1. Calculations of bearings and lengths of very short lines can give spurious results.
2. Perturbations of the object, due to rounding errors or changes of coordinate system can cause topological failures of a more serious nature – e.g. failures of axiom A10.

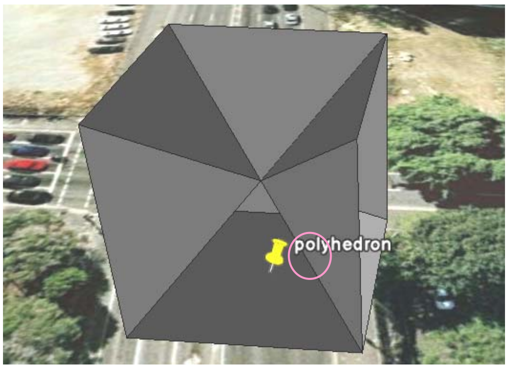


Figure 5 A failure of Axiom A1. The points within the circled area are about 4mm apart. (The front and top faces have been made transparent)

Axiom A3 The faces incident at a node do not intersect one another except at an edge.

$$\forall n \in s\colon \forall f_1, f_2 \in s,\ n \in f_1,\ n \in f_2;\ \mathrm{On}(p, f_1),\ \mathrm{On}(p, f_2),\ p \neq n$$
$$\Rightarrow \exists\ e\colon e \in f_1 \wedge e \in f_2,$$

Violation of this axiom potentially makes it impossible to determine “inside” and “outside” of an object. See the top prism of Figure 6, which is “inside out”.

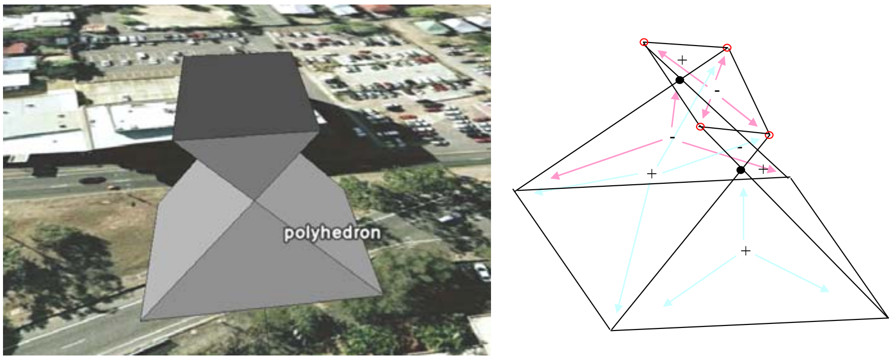


Figure 6. There is no edge between the points marked with black dots. The faces that cross between the dots are planar (and have 6 edges that define them). This has a genus of 0. Faces towards the viewer have a +, and are shown in blue, those that face away are shown in pink, with a -

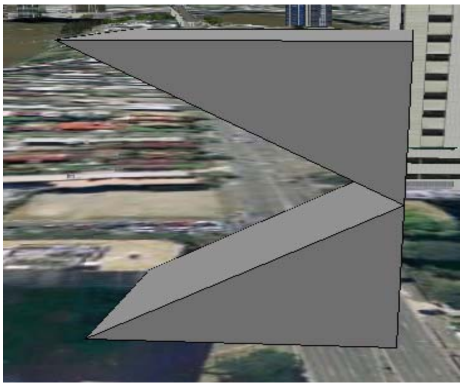


Figure 7. A more restricted case of failure of this axiom. The rightmost face does not have an edge which meets the diagonal faces, but does have nodes where they meet

Axiom A5 Non-intersecting edges must not be within a distance ε of each other.

A5. $\forall e_1, e_2 \in s, D(e_1, e_2) < \varepsilon \Rightarrow \exists\, n: n \in e_1, n \in e_2$.

There are two test cases shown, Figure 8, in which the end of one edge is near the other, and Figure 9 where the interior of the edges are close. Failure to observe this axiom means that perturbations of the object can cause topological failures of a more serious nature.

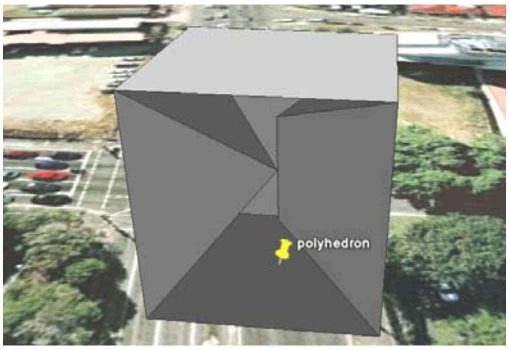


Figure 8. The central node is 9mm from the nearby edge. (The front face has been made transparent)

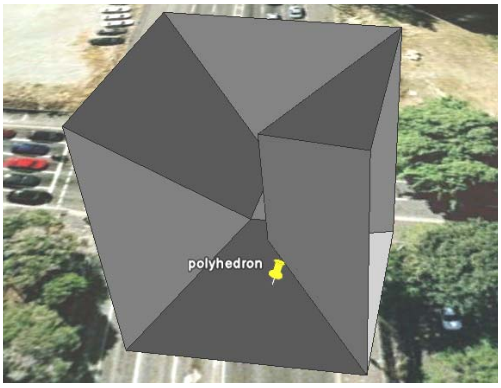


Figure 9. The edges are 1.5mm apart in the centre of the object. (Front and top faces made transparent)

Axiom A6 Every directed-edge of a face in the shell, belongs to a fold.

(That is, each edge is the meeting of an even number of faces such that the directed-edges of those faces form an alternating sequence around the edge).

A6. $\forall e \in s, \exists\, v \in s: e \in v$.

Failure of this axiom can mean that the alternating sequence of faces is not observed and the folds are broken. (Figure 10) Objects which violate this axiom may not have a well-defined interior.

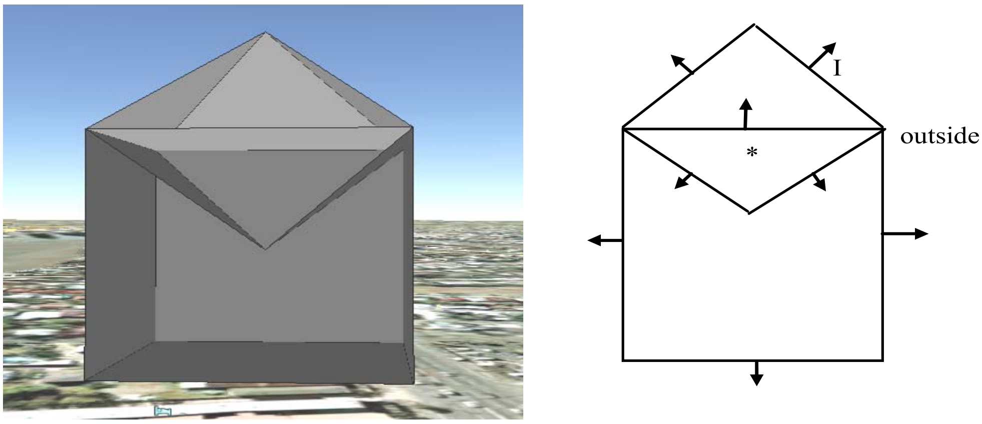


Figure 10. Front faces made transparent, right – a cross section of the parcel. This object is in effect the intersection of a cube and a isohedron, The point marked "*" in the diagram is within both, and therefore "twice within" the object

Axiom A8 Bounded faces are planar to a tolerance of ε'.

A8. $\forall f \in F, \forall\ n \in f, \mathrm{D}(n, f_p) \leq \varepsilon'$.

Any surface that is based on a polygon of more than 3 points may exhibit this failure. If the failure is extreme, the interior of the parcel is not legally definable.

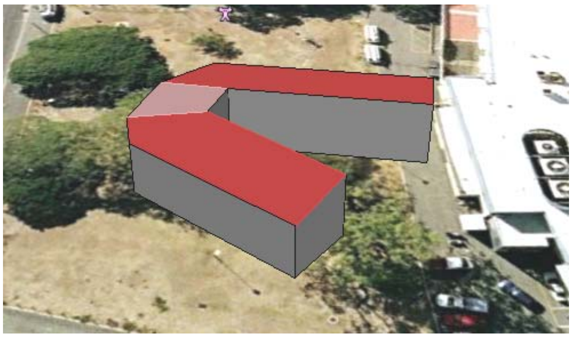


Figure 11. This test case is based on a real cadastral parcel, which contains surfaces which are out of plane. The darker coloured surfaces are 140 and 227 mm out of plane, the pink surface is 6.7mm out of plane

Axiom A9 No node is within ε of a face unless it is part of the definition of that face.

A9. $\forall f, n \in s, n \notin f \Rightarrow \neg \mathrm{In}(n, f) \vee \mathrm{D}(n, f_p) > \varepsilon$.

Failure to observe this axiom means that perturbations of the object could cause topological failures of a more serious nature.

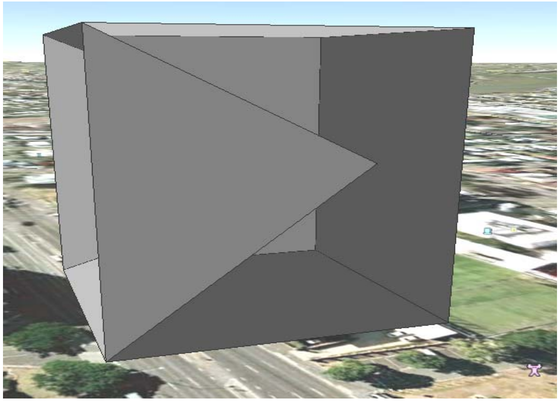


Figure 12 The point is 1.5 mm from the face on the right. (Note there is no vertex in the middle of the face on the right) The front face has been made transparent

Axiom A10 No directed-edge intersects a face except at a node of that edge.

$$\forall e \in s, \forall f \in s: e \notin f,$$

$$(\exists\, p: \mathrm{D}(p, e) = 0 \wedge \mathrm{D}(p, f) = 0) \Rightarrow p = n_1 \vee p = n_2, \text{ where } e = (n_1, n_2).$$

An object which violates this test has no unambiguous definition of inside/outside.

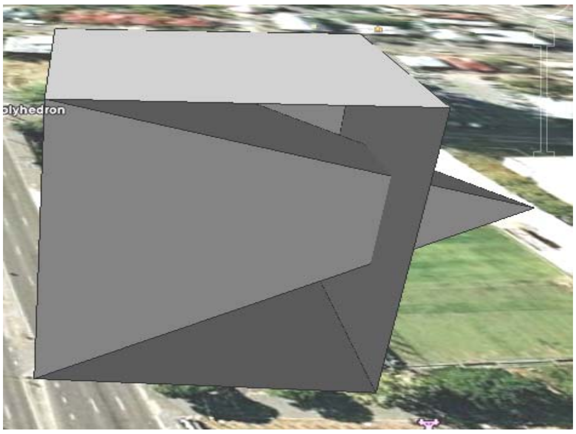


Figure 13. This is a more extreme case than Figure 12, and indicates what may occur if that case is perturbed. The front face has been made transparent

3.2 Parsimony axioms

These axioms might be considered “nice to have”. Failure to respect these axioms may not have extreme consequences such as those above, but may lead to excessive storage and difficulty in maintenance. In some cases there may be good reasons to violate these axioms, as noted below.

Axiom A7 The semi-edges that delineate a hole in a face must be part of the outer boundary of other faces.

A7. $\forall n \in s, \exists f_1, f_2 \in s: n \in f_1, n \in f_2, f_1 \neq f_2$.

(For any node, there must be at least two distinct faces that it participates in)

There are no severe consequences that follow from violating this axiom, and the axiom is redundant if axiom A2 is mandated (A2 ⇒ A7).A minor consequence is that if the total length of the edges of a parcel is calculated, the unnecessary edges will be counted. On the other hand, as shown in (Thompson and Van Oosterom 2011), a 2-shell subset of a 3D parcel may violate this axiom.

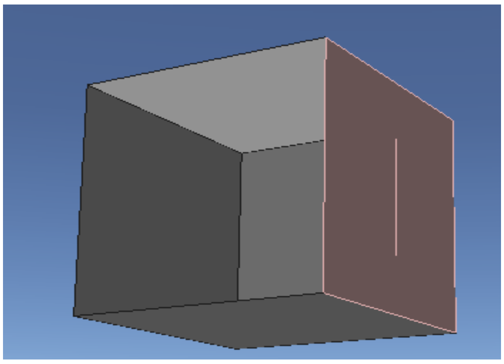


Figure 14. The line on the right face indicates a "hole" in the face, defined by a pair of edges. (The front face has been made transparent)

Axiom A2 Each node has at least 3 incident faces.

A2. $\forall n \in s, \exists f_1, f_2, f_3 \in s: n \in f_1, n \in f_2, n \in f_3\ f_1 \neq f_2 \wedge f_2 \neq f_3 \wedge f_1 \neq f_3.$

This is a deceptively simple axiom. On the one hand, it prevents unnecessary points, but on the other it may be violated by separating a compliant cycle shell into C_2 connected sub-shells (Thompson and Van Oosterom 2011). That paper states that:
If P is a 3D parcel which satisfies axioms A1 to A10 (but which may be discontinuous):
Any 0-shell of P obeys all axioms A1 to A10.
Any 1-shell of P obeys all axioms A1 to A10 except A2.
Any 2-shell of P obeys all axioms A1 to A10 except A2 and A7.
This may be seen as a strong case to omit axioms A2 and A7.

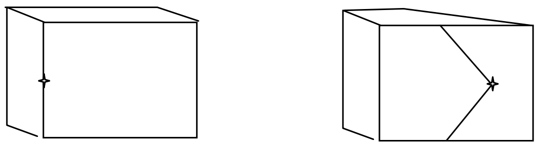
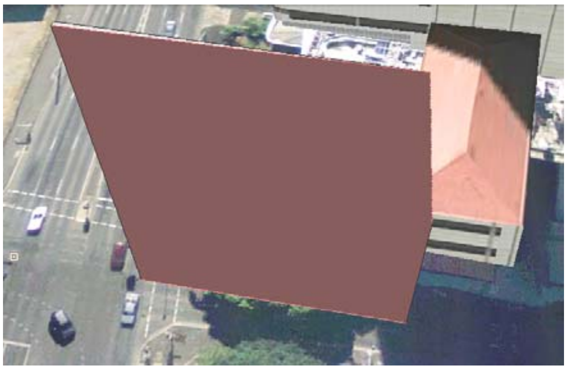


Figure 15. Shapes that do not respect axiom A2

On the other hand, it is possible to construct an example that violates no axioms except A2, but which has an unfortunate appearance (Figure 16). It is not expected that two planes should meet at a bent line. Since this case is a consequence of the extremely fine dihedral angle of these faces, a solution to this could be to ban faces that meet at an angle less than some limit. Sharp wedges can have other unpleasant consequences.



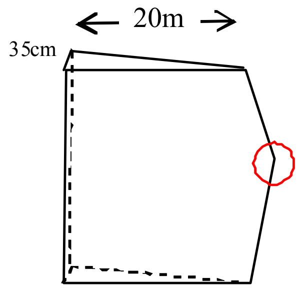


Figure 16. The object is like a thin wedge of cheese, 35 cm at the widest. The two largest faces are almost planar (worst case 5.6mm out of plane), but the circled vertex is 1m off the straight line

Axiom A2a No neighbour faces in same plane.

There are a number of other cases of unnecessary complication in a parcel definition.

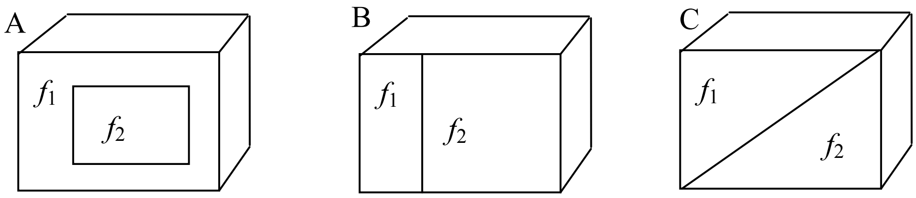


Figure 17. In all of these cases, f_1 and f_2 are coplanar, and together comprise the front face of the object

For example, in Figure 17 A, the face f_2 fills in a rectangular hole in f_1. If it were removed, and the hole filled in, the point-set definition of A would not change.

These cases could be prevented by a blanket rule that "adjoining faces must not be coplanar" (to a tolerance). As can be seen in Figure 17 C, this would prevent the unnecessary triangulation of surfaces, which may not be a good thing. In Queensland, there is a convention developing that vertical or horizontal surfaces are represented as polygons and are by definition flat, while all other surfaces are triangulated (Figure 18). It may well be that some of the adjacent faces in Figure 18 are nearly coplanar and may even violate A2a at a given tolerance.

Figure 18. Sections of a tunnel parcel. The top surface is not horizontal, and has been triangulated. The sides are vertical and are composed of rectangles. (The z values have all been increased by 40m so that it is visible above ground)

Axiom S1. No face may be paired with an anti-equal face in the same shell. (Disallowing dangling faces, but also disallowing the interior to be cut by a face.)

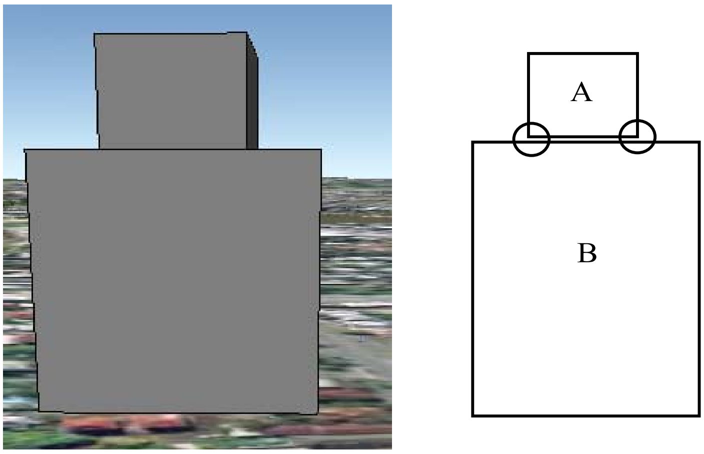


Figure 19. There is a pair of anti-equal faces between parts A and B

An anti-equal pair of faces such as this may make the shell fail the test of being a 2-shell (as in Figure 19), while it has all the appearances and real-world characteristics of a 2-shell. If there is some distinction between parts of a shell that need to be highlighted by anti-equal pairs of faces, then there will almost certainly be attribute differences between them.

3.3 Redundant axiom

The axiom A4 in an earlier paper (Thompson and van Oosterom 2012) has proved to be redundant. In that paper A4 stated: "Edges do not intersect except at their common end point nodes". It is now clear that A5 $\Rightarrow$ A4.

Proof; If e_1, e_2 intersect at point p, then $D(e_1, e_2) = 0$.
By A5, $D(e_1, e_2) < \varepsilon \Rightarrow \exists n: n \in e_1, n \in e_2$.
Since straight lines can only intersect at one point, $p = n$, i.e. the edges meet at common end point nodes.

4. LAND ADMINISTRATION DOMAIN MODEL

The representations of parcels in ISO19152 (The Land Administration Domain Model – LADM) (ISO-TC211 2009) is specified at five different levels of encoding, but it is only at the two higher levels that any serious validation is possible – the "polygon based" and the "topology based" levels. The standard also allows 2D, 3D and mixed dimensionality parcels. As observed by (Stoter and van Oosterom 2006), the key to mixing 2D and 3D cadastral parcel is to recognise that what is commonly referred to as 2D is in fact the representation of a 3D space that extends above and below the defined parcel.
The LADM handles this mixture by defining two boundary objects – the LA_BoundaryFaceString (used mainly for 2D parcels, but which may abut or surround a 3D parcel), and the LA_BoundaryFace (used for 3D parcels) (Lemmen, Van Oosterom, Thompson, Hespanha and Uitermark 2010). The concept of the LA_BoundaryFaceString

(here referred to as FaceString) is that it is defined by a simple linear GM_Curve, but interpreted as a series of faces which run from an undefined negative height to an undefined positive height (notated as $-\infty$ and ∞). See Figure 20

4.1 Polygon based spatial units

This section is a discussion of how these validation rules can be applied to parcels that are encoded according to the LADM profile of the polygon based spatial units – that is encoded as separate parcels without topology being specifically encoded. Spatial units may be 2D or 3D or any combination, and may have faces in any orientation (not merely vertical or horizontal).

In this form, each spatial unit is a separate polygon (or polyhedron). There are restrictions on the overlap of spatial units, but these are not imposed by the data model (as they are in the case of the Topology Based encoding). It would be common for a 3D object to be within the space defined by a 2D parcel in this approach, and overlaps of different interests are common (e.g. easements over base parcels).

In this model, the FaceString is considered to be a set of faces, each being defined by a single linear segment. Thus the segment from point (x_1,y_1) to (x_2,y_2) in the linestring that defines the FaceString is considered as a face defined by four points $(x_1,y_1,-\infty)$, $(x_2,y_2,-\infty)$, (x_2,y_2,∞), (x_1,y_1,∞). See Figure 20.

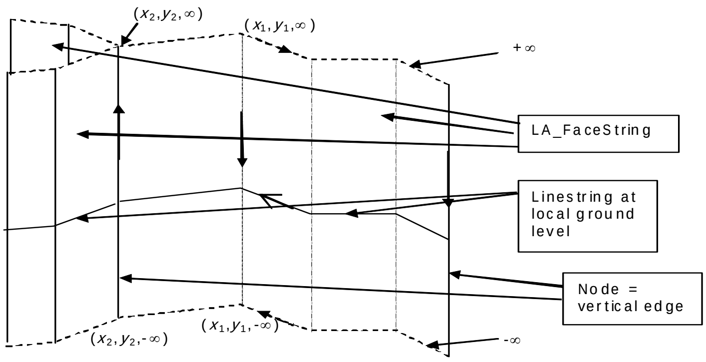


Figure 20. Segment of a FaceString interpreted as a face. (Note - the direction of this face is clockwise because it is being viewed from inside the parcel it defines)

An LA_BoundaryFace is simply treated as a face, and as noted before need not be vertical or horizontal. In the case of faces that extend to infinity, but the following restrictions apply:

$\forall e = ((x_1,y_1,z_1), (x_2,y_2,-\infty)),\ z_1 \neq -\infty$: $x_1 = x_2$ and $y_1 = y_2$.

$\forall e = ((x_1,y_1,z_1), (x_2,y_2,\infty)),\ z_1 \neq \infty$: $x_1 = x_2$ and $y_1 = y_2$.

$\forall e = ((x_1,y_1,-\infty), (x_2,y_2,z_2)),\ z_2 \neq -\infty$: $x_1 = x_2$ and $y_1 = y_2$.

$\forall e = ((x_1,y_1,\infty), (x_2,y_2,z_2),\ z_2 \neq \infty$: $x_1 = x_2$ and $y_1 = y_2$.

(That is, every edge running to or from $\pm\infty$ must be vertical. Otherwise, it has an undefinable slope).

Given these interpretations, a SpatialUnit can be interpreted as a shell of faces as defined here, with top and bottom faces defined from the GM_Curves (setting the z values to $\pm\infty$ and reversing the sense of the top face). Constraints would usually be placed on this that it be a cycle shell, and that it satisfy the core axioms. It may be also mandated that a SpatialUnit be a 2-shell, and that any of the parsimony axioms should apply as decided by the jurisdiction.

4.2 Topology based spatial units

This section is a discussion of how these validation rules can be applied to parcels that are encoded according to the LADM profile of the topology based spatial units. Spatial units may be 2D or 3D or any combination, and may have faces in any orientation.

In this form, the set of spatial units forms a linear partition of space. The overlap of spatial units is prevented by the data model, except where layering is used.

In this model, each linear segment of the FaceString is considered to be a pair of anti-equal faces. Thus the segment from point (x_1,y_1) to (x_2,y_2) in the linestring that defines the FaceString is considered as a face defined by four points $(x_1,y_1,-\infty)$, $(x_2,y_2,-\infty)$, (x_2,y_2,∞), (x_1,y_1,∞), paired with a face defined by the four points (x_1,y_1,∞), (x_2,y_2,∞), $(x_2,y_2,-\infty)$, $(x_1,y_1,-\infty)$. See Figure 21.

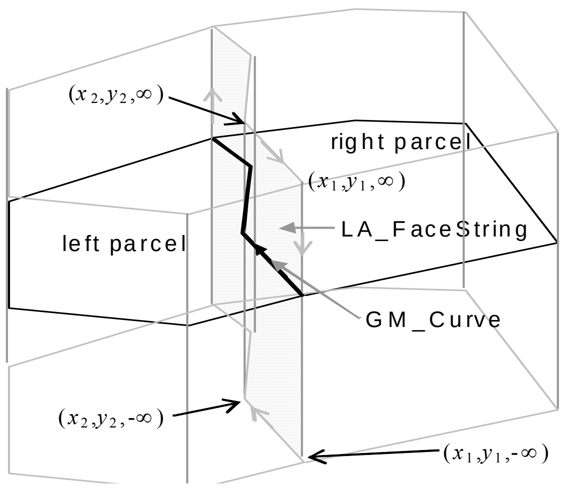


Figure 21. A segment of a FaceString interpreted as a pair of faces

An LA_BoundaryFace is treated as a pair of faces, with the same restriction as above:

$\forall e = ((x_1,y_1,z_1), (x_2,y_2,-\infty)), z_1 \neq -\infty$: $x_1 = x_2$ and $y_1 = y_2$.

$\forall e = ((x_1,y_1,z_1), (x_2,y_2,\infty)), z_1 \neq \infty$: $x_1 = x_2$ and $y_1 = y_2$.

$\forall e = ((x_1,y_1,-\infty), (x_2,y_2,z_2)), z_2 \neq -\infty$: $x_1 = x_2$ and $y_1 = y_2$.

$\forall e = ((x_1,y_1,\infty), (x_2,y_2,z_2), z_2 \neq \infty$: $x_1 = x_2$ and $y_1 = y_2$.

As for the polygon based encoding, top and bottom faces are based on the GM_curves.

Given these interpretations, the entire set of spatial units may be interpreted as a cycle shell as defined above, which satisfies the core axioms but does not satisfy axiom S1. It may be also mandated that this set be a 2-shell, and that any of the parsimony axioms should apply as decided by the jurisdiction.

In addition, each individual spatial unit can be interpreted as a cycle shell, which again must satisfy the core axioms, and as many of the remaining axioms as the jurisdiction should decide.

Note that this approach defines a shell, which does not correspond to any spatial unit in the set. This is the outer boundary of the entire set. Again, this can be mandated to be a 2-shell if appropriate.

5. CONCLUSIONS AND FUTURE RESEARCH

5.1 Conclusions

A set of definitions and axioms has been developed to be applied to cadastral data, and these have been shown to address two problems, which correspond to two of the spatial unit encodings defined in ISO19152:

1. The modelling of a single cadastral parcel (a Polygon Based Spatial Unit).
2. The modelling of a linear partition of space (the Polygon based Spatial Units).

5.2 Further Research

Although the axioms presented here have been shown to be independent and necessary for "well behaved" cadastral parcels, there may well be other validation tests possible that can detect ill formed cadastral spatial units that would be accepted by these. This will always be the case, since it is not possible to guess all ways that incorrect data can be encoded. It is, however possible to add to this list, provided that any additional axioms or tests should be subjected to the same scrutiny. The question should be asked of any other proposed tests:

What is the cost of accepting data that fails this test?

Using these definitions and axioms, many other results follow, some trivial, others not so (such as for a Cycle Shell with no points at infinity, a volume can be calculated). As many as possible of these results need to be catalogued, and proven. The behaviour of the faces at ± infinity, and of the outer boundary set of a set of topology based spatial units can be further investigated.

Java routines have been written to verify that each of the axioms is individually testable. These are not sophisticated or efficient, especially as each axiom has been tested as a single stand-alone procedure (to ensure that no other axioms are also violated). For example, it is easy to modify the test for "isCycle" for a shell so as to trap violations of A6 and S1, but this was carefully avoided in the test routines. For a practical implementation, more efficient algorithms could result from combining many axioms into a common routine (although it would be preferred that in the case of an error, the correct axiom is reported).

ACKNOWLEDGEMENTS

This research is supported by the Dutch Technology Foundation STW, which is part of the Netherlands Organisation for Scientific Research (NWO) and partly funded by the Ministry of Economic Affairs, Agriculture and Innovation (Project codes: 11300 and 11185).

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BIOGRAPHICAL NOTES

Rod Thompson has been working in the spatial information field since 1985. He designed and led the implementation of the Queensland Digital Cadastral Data Base, and is now principal advisor in spatial databases. He obtained a PhD at the Delft University of Technology in December 2007.

Peter van Oosterom obtained an MSc in Technical Computer Science in 1985 from Delft University of Technology, The Netherlands. In 1990 he received a PhD from Leiden University for this thesis 'Reactive Data Structures for GIS'. From 1985 until 1995 he worked at the TNO-FEL laboratory in The Hague, The Netherlands as a computer scientist. From 1995 until 2000 he was senior information manager at the Dutch Cadastre, were he was involved in the renewal of the Cadastral (Geographic) database. Since 2000, he is professor at the Delft University of Technology (OTB Research Institute for the Built Environment) and head of the GIS Technology Section. He is the current chair of the FIG joint commission 3 and 7 working group on '3D Cadastres' (2010-2014).

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